Stabilization of Optimum Trajectory Costate Differential Equations

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Introduction

IN trajectory optimization and synthesis of initial-value and two-point boundary-value guidance and control laws, most applications of the Euler–Lagrange first-variation necessary conditions of the calculus of variations have led to asymptotically unstable costate differential equations and, hence, unstable behavior of system physical states. This instability has been most troublesome when solving for long-duration extremal arcs, particularly when significant state-dependent nonlinear forces act on the flight vehicle. For example, the purpose might be to design minimum fuel consumption trajectories for cruising flight of a high L/D aircraft; the optimum solutions may involve sustained phugoidal oscillations, but that behavior can be obscured by rapid costate divergence.

This Engineering Note introduces a method for stabilizing costate differential equations via analytical phase adjustment of the time-varying costates. The method employs a time-varying phase-adjustment parameter. For a given model of the vehicle dynamics, a necessary lower bound on the phase-adjustment parameter is derived, and a stronger sufficient condition is suggested. For simplicity, the subject is here presented in the context of a first-order one-dimensional system, but extension to multidimensional systems is straightforward.

Consider first a one-dimensional guided object (such as a sled or railcar) that moves horizontally on a straight path and has the linear, time-invariant state equations:

$$\dot{V} + a_1 V = bT \tag{1}$$

$$\dot{x} = V \tag{2}$$

where x is position, V is velocity, T is thrust (positive in the direction of positive V), a_1 is a constant coefficient of friction ($a_1 > 0$), and b is a constant reciprocal of mass ($b > 0$). Let the criterion of performance that is to be minimized by a positioning and/or velocity guidance system be

$$\int_{t_0}^{t_f} F_1(T, V, \dot{V}, \dot{x})\,\mathrm{d}t$$

where t_0 and t_f are initial and final times and

$$F_1 = -\Lambda V^2 + T^2/K + \lambda_1(\dot{V} + a_1 V - bT) + \lambda_2(\dot{x} - V) \tag{3}$$

Λ and K are constant Lagrange multipliers, and λ_1 and λ_2 are, in general, time-varying Lagrange multipliers (costates) used to introduce their respective anholonomic subsidiary conditions from Eqs. (1) and (2).

The Euler–Lagrange first-variation necessary conditions provide, for $t_0 \le t \le t_f$,

$$T = bK\lambda_1/2 \tag{4}$$

$$\dot{\lambda}_1 = a_1\lambda_1 - 2\Lambda V - \lambda_2 \tag{5}$$

$$\dot{\lambda}_2 = 0 \tag{6}$$

Therefore,

$$\dot{T} = a_1 T - bK(\Lambda V + \lambda_2/2) \tag{7}$$

wherein λ_2 is constant.

The time responses of the system comprising the plant of Eqs. (1) and (2) and the guidance law of Eq. (7) are

$$T = -[(bK\lambda_2/2)/\omega][\sin\omega t + (a_1/\omega)(1 - \cos\omega t)] \tag{8}$$

$$V = -[(bK\lambda_2/2)/\omega](1 - \cos\omega t) \tag{9}$$

$$x = x_0 - \left[\left(b^2K\lambda_2/2\right)/\omega^2\right][t - (1/\omega)\sin\omega t] \tag{10}$$

in which, if $\Lambda > -a_1^2/(b^2K)$, then $\omega = (a_1^2 + b^2K\Lambda)^{1/2}$, and the solution is oscillatory. To solve the two-point boundary-value (TPBV) guidance problem, it is convenient to take $t_f = t_0 + \pi/\omega$, providing a maneuver duration equal to one-half of the oscillatory period. Then, if $T_0 = T_f = 0$ and $V_0 = V_f = 0$, one has $x_f = x_0 - \pi b^2 K\lambda_2/(2\omega^2)$, so that

$$\lambda_2 = -\frac{2(x_f - x_0)\left(a_1^2 + b^2K\Lambda\right)^{\frac{3}{2}}}{\pi b^2 K} \tag{11}$$

With the solution structured in this way, oscillation is not of practical significance. Matters are not so simple, however, for linear time-varying (LTV) and nonlinear plants, which can exhibit rapid aperiodic or oscillatory divergence because of costate instability.

To see this, in place of Eqs. (1) and (2) next take the following nonlinear, time-invariant state equations, which are written with the assumption that a_2, V, and T are all positive semidefinite:

$$\dot{V} + a_2 V^2 = bT \tag{12}$$

$$\dot{x} = V \tag{13}$$

The a_2V^2 term is suggestive of air resistance acting on a flight vehicle. We use the criterion integrand

$$F_2 = -\Lambda V^2 + T^2/K + \lambda_1\left(\dot{V} + a_2V^2 - bT\right) + \lambda_2(\dot{x} - V) \tag{14}$$

Equations (3) (F_1) and (14) (F_2) are conventional criterion integrands; the quadratic guidance effort penalty, T^2/K, with $K > 0$, is consistent with Legendre second-variation tests, and the terms introduced by λ_1 and λ_2 pertain to the plant dynamics and kinematics. Traditionally, the $-\Lambda V^2$ term is used with positive Λ to engender high kinetic energy and enhanced aerodynamic controllability along the extremal path.

Using F_2, one again obtains Eqs. (4) and (6), but now the costate differential equation (DE) becomes

$$\dot{\lambda}_1 = 2a_2V\lambda_1 - 2\Lambda V - \lambda_2 \tag{15}$$

and, therefore, the guidance law is

$$\dot{T} = 2a_2VT - bK(\Lambda V + \lambda_2/2) \tag{16}$$

The system represented by Eqs. (12), (13), and (16) exhibits divergent behavior. The divergence can be so rapid that the usable time horizon is insufficient for practical application. Fortunately, divergence can be eliminated by phase adjustment of the costate DE, Eq. (15).

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Costate Phase Adjustment and Stability

In Eq. (15), replace λ_1 by a phase-adjusted costate variable, $\lambda_1 + \tau^* \dot{\lambda}_1$, where τ^* is a potentially time-varying parameter to be determined by the needs of stability. Thus,

$$\dot{\lambda}_1 = 2a_2 V\left(\lambda_1 + \tau^* \dot{\lambda}_1\right) - 2\Lambda V - \lambda_2 \tag{17}$$

Because $\lambda_1 = 2T/(bK)$, Eq. (17) becomes

$$\dot{T} = \frac{-2a_2 V T + bK(\Lambda V + \lambda_2/2)}{2a_2 V \tau^* - 1} \tag{18}$$

Thus, if

$$\tau^* = Q/(2a_2 V) \tag{19}$$

in which Q is a constant, one has the LTV thrust-command DE

$$\dot{T} = a(t)T + \phi(V) \tag{20}$$

wherein

$$a(t) = -\frac{2a_2 V}{Q-1} \tag{21}$$

$$\phi(V) = \frac{bK(\Lambda V + \lambda_2/2)}{Q-1} \tag{22}$$

Our interest is in the stability of Eq. (20). Consider, first, the $a(t)T$ term in this equation; it is known that $\dot{T} = a(t)T$ is stable if $a(t) < 0$ for any fixed $t > t_0$ and if $|\dot{a}(t)|$ is sufficiently small.[1] For positive V, $a(t) < 0$ for any fixed t if $Q > 1$, that is, if $\tau^* > 1/(2a_2 V)$. Q determines $|\dot{a}(t)|$; when Q is increased, $|\dot{a}(t)|$ may be made arbitrarily small. However, there can be a cost associated with use of Q values substantially greater than unity, because larger Q produces slower responses, requiring longer maneuver durations, $t_f - t_0$.

The stability thresholds on Q and $|\dot{a}(t)|$ are closely related; from Eq. (21)

$$\dot{a}(t) = -\frac{2a_2 \dot{V}(t)}{Q-1} \tag{23}$$

so that [see Eq. (12)]

$$\dot{a}(t) = -\frac{2a_2\left[bT(t) - a_2 V^2(t)\right]}{Q-1} \tag{24}$$

Provided $Q > 1$, thrust limiting may be employed to impose a maximum bound, $|\dot{a}_{\max}|$, on the magnitude of $\dot{a}(t)$. For nonnegative T, the thrust bounds (which open wider as Q is increased) are

$$T_{\max} = \left[a_2 V^2 + |\dot{a}|_{\max}(Q-1)\big/(2a_2)\right]\big/b \tag{25}$$

$$T_{\min} = \left[a_2 V^2 - |\dot{a}|_{\max}(Q-1)\big/(2a_2)\right]\big/b \tag{26}$$

Because of its inherent nonlinear character, thrust limiting may not be a good means for ensuring stability: Proper sizing of Q is generally preferable because of its linear action on the system.

We have seen that Q must exceed unity. One can also establish larger lower bounds on Q that satisfy the sufficiency requirement on $|\dot{a}(t)|$. As a first guide (in the TPBV problem), one may consider the average value of $\dot{V}$, that is, $\dot{V}_{\mathrm{av}} = (V_f - V_0)/(t_f - t_0)$, giving an average $\dot{a}$ during the maneuver of

$$\dot{a}_{\mathrm{av}} = -\frac{2a_2(V_f - V_0)}{(t_f - t_0)(Q-1)} \tag{27}$$

From simulation results to be presented,

$$t_f - t_0 \approx c_0 + c_1 \tau_f^* = c_0 + \frac{c_1 Q}{2a_2 V_f} \tag{28}$$

where c_0 and c_1 are ad hoc constants. Equation (27) now becomes

$$\dot{a}_{\mathrm{av}} \approx G_1/[(Q-1)(2c_0 a_2 V_f + c_1 Q)] \tag{29}$$

$$\dot{a}_{\mathrm{av}} \approx G_1/(AQ^2 + BQ + C) \tag{30}$$

in which

$$G_1 = -4a_2^2 V_f (V_f - V_0) \tag{31}$$

$$A = c_1 \tag{32}$$

$$B = 2c_0 a_2 V_f - c_1 \tag{33}$$

$$C = -2c_0 a_2 V_f \tag{34}$$

Thus,

$$|Q - R_1||Q - R_2| \approx |G_1|/|\dot{a}_{\mathrm{av}}| \tag{35}$$

where R_1 and R_2 are the roots of the quadratic expression in Eq. (30),

$$R_{1,2} = G_2\left(1 \pm \sqrt{1 + G_3}\right) \tag{36}$$

which uses the further definitions,

$$G_2 = \frac{1}{2} - \frac{c_0 a_2 V_f}{c_1} \tag{37}$$

$$G_3 = \frac{2c_0 a_2 V_f}{c_1 G_2^2} \tag{38}$$

Equation (35) may be solved for the larger of two Q values, given the prescribed $|\dot{a}_{\mathrm{av}}|$, or the relationships between Q and $|\dot{a}_{\mathrm{av}}|$ or between Q and the peak $|\dot{a}|$ can be established empirically via simulations (shown subsequently).

In the case of initial-value formulations, particularly for maneuvers of long duration, the system must also exhibit stable responses to changes in V. Linearizing Eq. (20) at the operating point (T, V), one obtains

$$\frac{\mathrm{d}(\overline{\Delta T})}{\mathrm{d}t} = \frac{(-2a_2 T + bK\Lambda)\overline{\Delta V} - 2a_2 V \overline{\Delta T}}{Q-1} \tag{39}$$

in which $\overline{\Delta T}$ and $\overline{\Delta V}$ are small perturbations in T and V, respectively. For long-term speed stability, it is necessary that $Q > 1$ and $(-2a_2 T + bK\Lambda) < 0$, that is, $T > bK\Lambda/(2a_2)$.

Costate stability is necessary but may not be sufficient in every case for system stability. In the present example, the system comprises the nonlinear, time-varying DEs of Eqs. (12) and (20). To verify the stability of this system, differentiate Eq. (12) to obtain

$$\ddot{V} + 2a_2 V \dot{V} = b\dot{T} \tag{40}$$

where, from Eq. (20)

$$\dot{T} = \frac{-2a_2 V T + bK(\Lambda V + \lambda_2/2)}{Q-1} \tag{41}$$

Substituting for T from Eq. (12), one has

$$\dot{T} = \frac{-2a_2 V\left(\dot{V} + a_2 V^2\right)\big/b + bK(\Lambda V + \lambda_2/2)}{Q-1} \tag{42}$$

and therefore

$$\ddot{V} = a(t)\left(Q\dot{V} + a_2 V^2\right) + \frac{b^2 K(\Lambda V + \lambda_2/2)}{Q-1} \tag{43}$$

Therefore, the total system of Eqs. (12) and (20) is made stable by satisfying the same conditions required for stabilization of its costate DE.

The costate phase-adjustment parameter τ^* has the dimension of time. For this application [the plant of Eqs. (12) and (13)], it is convenient to think in terms of a characteristic length, $\ell = \tau^* V = Q/(2a_2)$, that is being held constant. The costate phase adjustment uses a variable lead time τ^* that keeps ℓ fixed as V changes.

Simulation Results

Equations (12), (13), and (20–22) have been used in a simulation with the physica parameter values $a_2 = 2 \times 10^{-5}$ and $b = 2 \times 10^{-4}$.

Heun's integration method was employed with a time step of $\Delta t = 0.001$ s, tested by comparison with Euler integration using $\Delta t = 10^{-6}$ s. The value of K was 1.0.

Increasing-velocity and decreasing-velocity runs were made using $\Lambda = 0$ and Q values of 1.01, 1.1, 2, and 11. The boundary conditions for the increasing-V runs were as follows: start, $t_0 = 0$, $V_0 = 300$, and $T_0 = 9000$, that is, $\dot{V}_0 = 0$; and finish, t_f to be determined (TBD), $V_f = 500$, and $T_f = 25{,}000$, that is, $\dot{V}_f = 0$. For the decreasing V runs, the boundary conditions were as follows: start, $t_0 = 0$, $V_0 = 500$, and $T_0 = 25{,}000$, that is, $\dot{V}_0 = 0$; and finish, t_f TBD, $V_f = 300$, and $T_f = 9000$, that is, $\dot{V}_f = 0$. Additionally, constant-V runs were performed.

To achieve the prescribed final conditions for given Λ and Q, a manual search was made for the appropriate λ_2 (which was a different constant for each run). The manual search achieved a final velocity precision of 1% or better in each run. (An automated search could have been used to obtain greater precision had that been necessary.) During each run, at such time as $|V_f - V|$ tested less than or equal to $0.01V_f$, the value of t_f was recorded and the thrust was immediately set to the known a priori value of T_f, after which integration of the $\dot{V}$ equation proceeded until solution halt at 1000 s.

The $Q = 1.01$ runs exhibited the greatest thrust and $|\dot{a}|$ peaks.

Table 1 summarizes λ_2 and t_f vs Q for the $\Lambda = 0$ runs. Figures 1–3 show representative system time responses for the increasing-V cases. (The response for $Q = 1.01$ was virtually the same as shown for $Q = 1.1$.) Figure 4 shows trade curves of maneuver duration ($t_f - t_0$) vs $Q - 1$ for the increasing-V (lower curve) and decreasing-V

Table 1 Multiplier λ_2 and t_f vs Q for $\Lambda = 0$ runs

Q	$\lambda_2 \times 10^{-6}$	t_f
	Increasing V	
1.01	4.95	136
1.1	4.944	135
2	4.8	191
11	5.77	652
	Constant V, $V_0 = 300$	
N/A	1.08	N/A
	Constant V, $V_0 = 500$	
N/A	5.0	N/A
	Decreasing V	
1.01	1.091	252
1.1	1.092	250
2	1.1219	315
11	0.91	1392

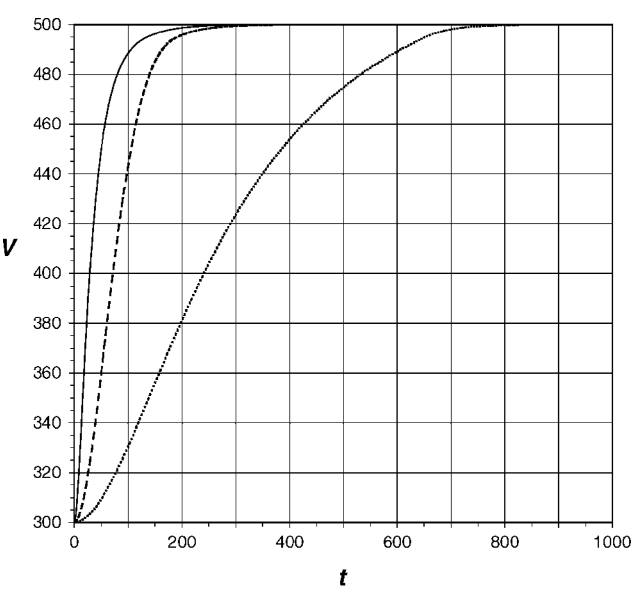

Fig. 1 Velocity vs time for $Q = 1.1$ (fastest response), 2, and 11.

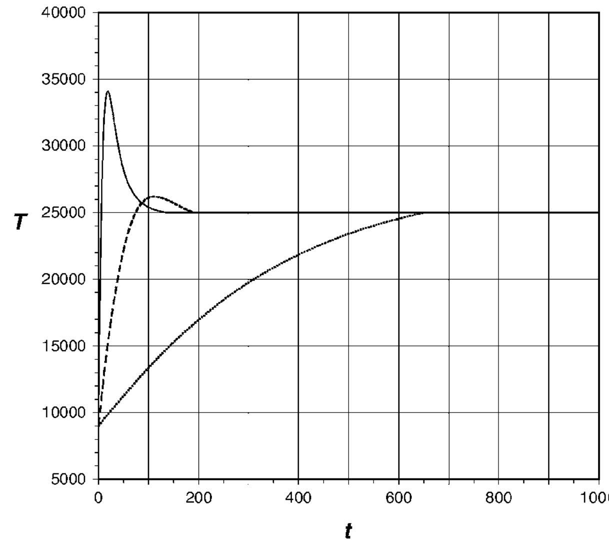

Fig. 2 Thrust vs time for $Q = 1.1$ (fastest response), 2, and 11.

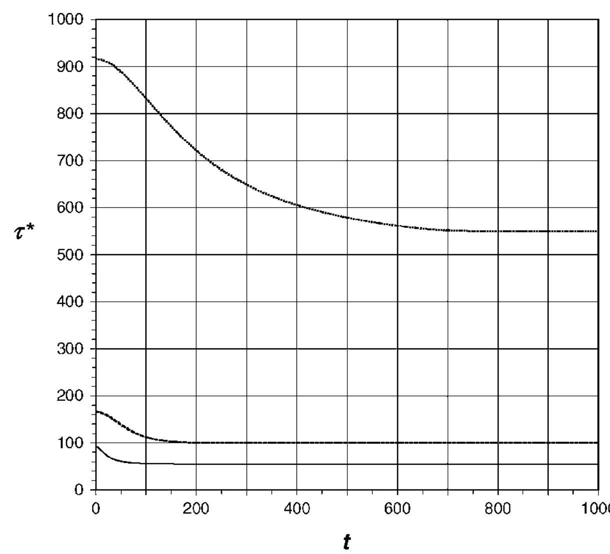

Fig. 3 Phase-adjustment parameter τ^* for $Q = 1.1$ (lowest curve), 2, and 11.

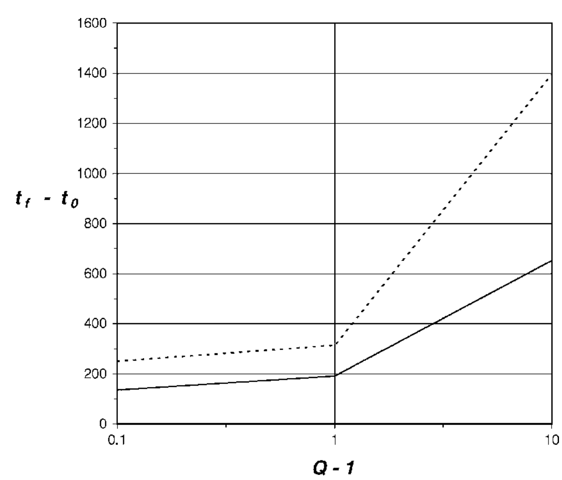

Fig. 4 Maneuver duration ($t_f - t_0$) vs $Q - 1$: ——, increasing-V solutions and – – –, decreasing-V solutions.

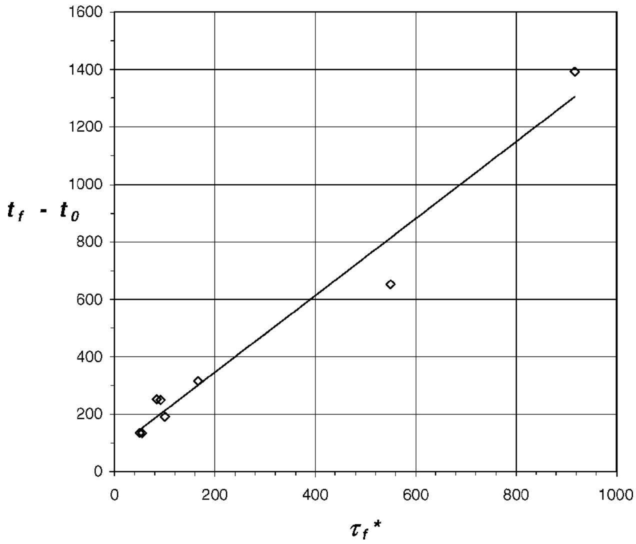

Fig. 5 Linear regression trendline for $(t_f - t_0)$ vs τ_f^*.

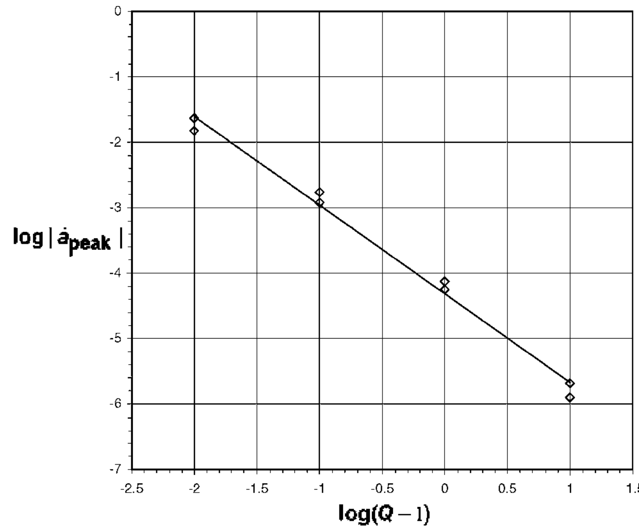

Fig. 6 Linear regression trendline for log$|\dot{a}_{\text{peak}}|$ vs log $(Q - 1)$.

solutions. Note that out to $Q = 2$ there is little sacrifice in response time.

Figure 5 shows the relationship between $t_f - t_0$ and τ_f^*. From Fig. 5 and regression analysis, it is seen that c_0 and c_1 in Eq. (28) are approximately 78.8 and 1.336, respectively, when data from all increasing-V and decreasing-V runs are treated. Separately, it was found that for the increasing-V runs that $(t_f - t_0) \approx 52(Q - 1)$ and for the decreasing-V runs that $(t_f - t_0) \approx 115(Q - 1)$.

Figure 6 presents a trendline relating peak $|\dot{a}|$ to $Q - 1$. The equation of this line (treating all data from increasing-V runs and decreasing-V runs) is

$$\log|\dot{a}_{\text{peak}}| = -1.354 \log(Q - 1) - 4.315 \tag{44}$$

Discussion

For short time-horizon TPBV solutions, costate asymptotic stability may not be required. An example is the receding-horizon optimal control formulation in which the value of $(t_f - t_0)$ may be as small as one sampling interval.[2] However for many TPBV problems, stability issues dominate, often eliminating the calculus-of-variations approach, unless tractable behavior of costates can be obtained. For initial-value problems, the costate solutions usually must be asymptotically stable.

This Engineering Note proposes a time-varying costate phase adjustment procedure suitable for both TPBV and initial-value formulations. For simplicity of presentation, a one-dimensional example is illustrated. For multistate systems, one should obtain the matrix characteristic equation (in terms of a single τ^*, or perhaps a separate τ^* for each state), and guarantee that the real parts of all of the eigenvalues have suitable polarity for a given fixed $t > t_0$ and that the same vary sufficiently slowly.

Conclusions

Costate DEs arising in calculus-of-variations optimization of guidance and control systems and flight vehicle trajectories can be stabilized via introduction of a costate phase-adjustment parameter. The stability boundary or boundaries on this parameter can be derived. If the phase parameter is only slightly greater than the value required for stabilization, there is little impact on system response time.

Acknowledgments

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